

# Possible-translations semantics for catholic modal logics

Juliana Bueno-Soler

GTAL/CLE and Department of Philosophy

State University of Campinas

P.O. Box 6133, 13083-970

Campinas, SP, Brazil

e-mail: `juliana.bueno@cle.unicamp.br`

## Abstract

This paper discusses the motivations and the possibilities for defining possible-translations semantics (introduced by W. A. Carnielli and further developed by W. A. Carnielli and J. Marcos) to a class of logics that we call *catholic modal logics*, defined as extensions of positive modal logics (here called *anodic modal logics*) by means of consistency (and inconsistency) operators. In this way, catholic modal logics are logics of formal inconsistency (the LFIs, as treated in [CCM07]) enriched with modal operators. The material here introduced constitutes first examples of modal systems endowed with possible-translations semantics. The motivations for studying catholic modal logics stem from the fact that these systems are able to provide alternatives for dealing with some modal paradoxes of diverse kinds, at least the ones engendered by the mixture of the (too strong) classical negation with modal operators.

## 1 Why catholic modalities

Our purpose is to obtain possible-translations semantics to several classes of modal paraconsistent logics. Modal paraconsistent systems seem to be very natural logical artifacts, while subsystems of modal logics that share with traditional modal logics several well-regarded logic features such as completeness, decidability, and so on. Modal paraconsistent logics have not been investigated as they deserve, and much less been taken into serious consideration by linguists and philosophers, perhaps not aware of the potentialities of such logics.

On a first overview, since its beginning, paraconsistent logics and modalities were not strange each other: already in 1948 S. Jaśkowski imagined a sort of modal environment to deal with contradictions observed in certain discourses (by means of his discussive or discursive logics). Indeed, Jaśkowski's discussive logic **D2** is a legitimate LFI, as argued in [CCM07] (example 93). However,

it was only in 1986, in [dCC86], that the first paraconsistent modal logic was proposed to deal with deontic paradoxes.

As recognized in the first lines of [vF73], “Since its inception deontic logic has been plagued by a series of paradoxes”. In that paper, van Fraassen tries to explain his “misgivings” about “orthodox” principles of deontic logics. He also refers to his previous work on conditional obligation, in [vF72], a notion he had introduced to handle deontic paradoxes and which leads itself to several other logical difficulties. It is part of my interest to investigate, in my Ph.D. Thesis (cf. [BS]), in which degree paraconsistent modalities could offer a good solution to some problems in conditional deontic logic.

Another line of attack to deontic paradoxes, by means of deontic modalities combined with logics of formal inconsistency, is developed in [Con07] and [PC], but paraconsistent modalities as a whole go far beyond deontic or norms: not only some problems described in [HPvdT07] can be thought in paraconsistent terms, but also certain problems and paradoxes in epistemic and doxastic logics gain a new insight when regarded from the paraconsistent perspective. For instance, Fich’s paradoxes of knowability is treated in terms of paraconsistent epistemic logics in [CL03]; other oddities of doxastic logics that can in principle be treated in paraconsistent terms are explained in [CP08], chapter 7.

In all cases, since the connection between modalities and logics of formal inconsistency are the constituent ingredients of such proposals, and considering how expressive the possible-translations semantics for paraconsistent logics are, it seems just natural to extend possible-translations semantics to modal paraconsistent logics. The intention of this note is to discuss this possibility. Attention is restricted to a particular class of systems, called  $\mathbf{mbC}^{k,l,m,n}$ , which are, at the same time, expressive enough to attend our motivations and complex enough to stimulate the here proposed definition of *modal possible-translations semantics*.

The reason we start from the logics of formal inconsistency (LFIs) is that such logics already dispose of the semantic machinery of possible-translations semantics. This kind of semantics is designed to permit us to understand the role of contradictory situations in argumentation, and also to explain the existence of conflicting scenarios without falling into deductive triviality.

In order to reach our purpose, we start from *modal extensions* of LFIs, that is, modal operators are added to the language of LFIs, and new axioms and rules are taken up according to the convenience and interest. These system are called *normal modal cathodic system* or simply *cathodic system*. A *minimal* version of the cathodic system  $\mathbf{S}$ , based on an LFI, is obtained by adding to  $\mathbf{S}$  the following:

- |              |   |                        |
|--------------|---|------------------------|
| <b>(K)</b>   | $\Box(p \supset q) \supset \Box p \supset \Box q$                               | (Axiom of <b>K</b> )   |
| <b>(Nec)</b> | $\vdash \alpha$ implies $\vdash \Box \alpha$                                    | (Necessitation rule)   |
| <b>(US)</b>  | $\vdash \alpha$ implies $\vdash \alpha[p/\beta]$ , where $p$ occurs in $\alpha$ | (Uniform substitution) |

The notation  $\alpha[p/\beta]$  means, as usual, the substitution of all occurrences of the variable  $p$  in  $\alpha$  by the formula  $\beta$ .

A full treatment on anodic and cathodic systems is reserved for [BS], where the aim is to characterize the class of cathodic systems starting from the implicative fragment of the propositional calculus **PC** (as proposed by Henkin in [Hen49]) and endowing the language with  $\wedge, \Box, \Diamond, \neg, \circ$ . Completeness results are obtained for each extension of the implicative fragment of **PC**. In that approach the particular system  $\mathbf{mbC}^{k,l,m,n}$  appears as a bi-modal system.

Since a form of classical negation can be defined in **mbC** (cf. [CCM07]), it turns out, as a consequence, that one can define the notion of “it is possible that” in terms of the notion of “it is necessary that”, as usual in modal logic. In fact,  $\Diamond\alpha$  and  $\sim\Box\sim\alpha$  become equivalent, and the resulting system becomes monomodal when we add to the system some adequate bridge principles, in the sense of [CC07]. In this way it is possible to show (details in [BS]) that both presentations of  $\mathbf{mbC}^{k,l,m,n}$ , as bi-modal and as monomodal logics, are equivalent (modulo certain bridge principles).

A hierarchy of normal modal cathodic systems is defined by adding to a minimal cathodic system instances of the axiom schema  $\Diamond^k\Box^l\alpha \supset \Box^m\Diamond^n\alpha$ , denoted by  $\mathbf{G}^{k,l,m,n}$ . This axiom schema, proposed by Lemmon and Scott in [LS77], is able to express several familiar modal axioms in the literature as for example  $\Box\alpha \supset \alpha$ , usually called **T**, by considering the instance  $k = m = n = 0$  and  $l = 1$ .

The usual Kripke semantics for modal logics are defined from frames  $\mathfrak{F} = \langle W, R \rangle$ , where  $W \neq \emptyset$  and  $R \subseteq \wp(W \times W)$  is an accessibility relation. To each modal axiom adopted in a given modal system, we associate an appropriate property of the accessibility relation  $R$  of  $\mathfrak{F}$ . For each instance of the axiom schema  $\mathbf{G}^{k,l,m,n}$  there is a correspondent property, noted by  $\mathbf{P}^{k,l,m,n}$  and called *convergence property*, which can be defined in first-order language. To illustrate this property consider the following conventions:

1.  $w_i R^0 w_j$  means that  $w_i = w_j$ ;
2.  $w_i R^m w_j$  denote that  $w_j$  is accessible from  $w_i$  in  $m$ -steps, i.e., there exists  $m - 1$  worlds  $w_{i+1} \cdots w_{i+(m-1)}$  such that  $w_i R w_{i+1} \cdots w_{i+(m-1)} R w_j$ .

In first-order language the property  $\mathbf{P}^{k,l,m,n}$  is expressed by:

$$\forall w_1 \forall w_2 \forall w_3 ((w_1 R^k w_2 \wedge w_1 R^m w_3) \supset \exists w_4 (w_2 R^l w_4 \wedge w_3 R^n w_4))$$

The convergence property  $\mathbf{P}^{k,l,m,n}$  generalizes the common properties associated to standard modal axioms; for instance, in the case of the axiom **T**, the relation  $R$  is reflexive. Each cathodic system  $\mathbf{mbC}^{k,l,m,n}$  can be shown to be sound and complete (cf. [BS]) with respect to a modal valuation semantics as defined in Definition 2.2. Our purpose here is to enrich the Kripke semantics with possible-translations semantics with the objective to provide a new interpretation to the cathodic systems. The reasons to attempt such an interpretation are given in next section.

## 2 The catholic systems $\mathbf{mbC}^{k,l,m,n}$ : a paradigmatic study case

A detailed treatment of LFIs was given by W. A. Carnielli, M. E. Coniglio and J. Marcos in [CCM07], where the authors investigate many examples of logics that express the notions of *consistency* and *inconsistency* as primitive symbols in the signature. We particularly focus the attention on the systems  $\mathbf{mbC}$ ,  $\mathbf{bC}$  and  $\mathbf{Ci}$ , treated in [CM02] and in [CCM07]. In our case, the correspondent catholic system of  $\mathbf{mbC}$ ,  $\mathbf{bC}$  and  $\mathbf{Ci}$  are denoted, respectively, by  $\mathbf{mbC}^{k,l,m,n}$ ,  $\mathbf{bC}^{k,l,m,n}$  and  $\mathbf{Ci}^{k,l,m,n}$ . Here, only the system  $\mathbf{mbC}^{k,l,m,n}$  will be specifically discussed, but extensions for the other systems are analogous<sup>1</sup>.

It is relevant to recall here two remarkable results involving LFIs and modal systems: neither of these types of logics can be characterized by finite matrices: the impossibility for the case of LFIs is carefully discussed in [CCM07], and for normal modal logics this is the well-known limitative result of J. Dugundji (see, for instance, [CP08]). This obviously means that any catholic system fails to be characterized by finitely-many valued semantics. So, besides better explaining the role of contradictions in argumentation, as mentioned before, the possible-translations semantics for catholic systems would be the closest possibility of assigning a (truth-functional) many-valued interpretation for such systems. This fact is one of the strongest motivations for dealing with possible-translations semantics for catholic modal logics.

Modal extensions of non-standard logics have been treated by many logicians; for example, M. Fitting dealt with modal extensions of many-valued logics in [Fit91] and [Fit92], but the particular case of modal extensions of three-valued logics was treated by K. Segerberg in [Seg67] and others. This kind of extension interest us here since we can use them to extend the notion of possible-translations semantics (PTS) to the catholic modal system. This is crucial because in the majority of cases the so-called *traducts* (i.e., the classes of logics able to provide a PTS to a given system) are many-valued logics; indeed, in our examples, just 3-valued logics.

We will show that this kind of modal extensions of many-valued logics provide a precise setting for the *modal possible-translations semantics*, and are particularly significant in the case of three-valued modal logics, as they endow quite natural interpretations for catholic logics.

The term *catholic* comes from chemistry, and refers to a negatively charged electrode. By analogy, the catholic systems are extensions of positive modal logics (systems that have only the connective of implication ( $\supset$ ) and the necessity operator ( $\Box$ ) in their language) by adding negations. Positive modal systems will be called *anodic*, exploiting the same chemical analogy.

The main characteristics of catholic systems is that their negations satisfy less properties than classical negation. The LFIs are precisely the logics that permit us to deal with weaker negations combined with operators of consistency

---

<sup>1</sup>A detailed investigation on some hierarchies of catholic systems is made in my Ph.D. Thesis (work in progress).

(denoted by  $\circ$ ) and inconsistency (denoted by  $\bullet$ ). The explosive character of these systems can be controlled by means of “consistent formulas” (formulas preceded by  $\circ$ ) in the sense that the trivialization is obtained from contradictory formulas plus its consistent content, but not without such consistent content. So, contradictions are now divorced from trivialization, and this can be very pertinent when modal reasoning is involved.

Our interest is focused in three particular systems: **mbC**, **bC** and **Ci**. The first one is interesting because it is the minimal system of the hierarchy whose language possesses the consistency operator, and the last one because it is the first system where the inconsistency operator can be defined within the language. **bC** is an intermediary system between **mbC** and **Ci**.

In [BS07] anodic systems have been approached from the point of view of quantified propositional modal logics, which would avoid a bi-modal language. That approach, however, is much harder to be generalized, and we abandoned it in favor of a more natural construction as explained below.

Our new construction (adopted in [BS]) starts from the implicative fragment of the standard propositional calculus, treated by L. Henkin in [Hen49], adding to it **(K)**, **(Nec)**, **(US)** called  $\mathbf{K}^\supset$ . In order to add the axiom schema  $\mathbf{G}^{k,l,m,n}$  to  $\mathbf{K}^\supset$  we need first to extend its language with  $\wedge$ , adding the known axioms for conjunction. The next step is to extend the language with  $\diamond$ , plus the following axioms:

$$\mathbf{(K1)} \quad \Box(p \supset q) \supset (\diamond p \supset \diamond q)$$

$$\mathbf{(K2)} \quad \diamond(\alpha \vee \beta) \supset \diamond\alpha \vee \diamond\beta$$

$$\mathbf{(K3)} \quad \diamond\alpha \supset \Box\beta \supset \Box(\alpha \supset \beta)$$

These axioms are instances of **(K)** in the usual modal systems. In this way we obtain a bi-modal anodic system. These extensions are called *normal modal anodic systems*, or simply, *anodic systems*.

Soundness and completeness can be obtained for these bi-modal cathodic systems with respect to the class of frames where the accessibility relation satisfies the convergent property corresponding to  $\mathbf{G}^{k,l,m,n}$ . Since those are bi-modal systems we must add two instances of  $\mathbf{G}^{k,l,m,n}$  so as to obtain completeness – the dual instance  $\mathbf{G}^{m,n,k,l}$  must be added together with  $\mathbf{G}^{k,l,m,n}$ . The construction of the canonical model, using certain “positive valuations”, is based on maximal prime sets (that is, non-trivial maximal sets that satisfy the condition:  $\Delta \vdash \alpha \vee \beta$  implies  $\Delta \vdash \alpha$  or  $\Delta \vdash \beta$ ) by means of constructions *à la* Lindenbaum. The completeness is restricted to *factual deductions* in the following sense:

1. A set of formulas  $\Delta$  is *factual* if there is a formula  $\alpha$  such that  $\diamond\alpha \notin \Delta$ , otherwise is called *hypothetical*.
2. A *factual deduction* is a derivation  $\Gamma \vdash \alpha$ , where  $\Gamma$  is a factual set.

The *cathodic systems* evolve from the anodic ones by parsimoniously adding negations, in such a way as to obtain a graded hierarchy of modalities combined

with negations. The basic cathodic system is the logic  $\mathbf{mbC}^{k,l,m,n}$ , and will be taken as a paradigmatic study case for defining modal possible-translations semantics.

The logic  $\mathbf{mbC}$  is strong enough to be able to express what is called a *supplementing negation*, defined by  $\wr\alpha \stackrel{\text{def}}{=} (\circ\alpha \wedge \neg\alpha)$ . So, a *classical negation* turns out to be defined in  $\mathbf{mbC}$  as  $\sim\alpha \stackrel{\text{def}}{=} \beta \supset (\beta \wedge \wr\beta)$ , for a fixed  $\beta$ . In [CCM07] it is shown that supplementing negations and classical negations are not necessarily equivalent. For our purposes, it is now convenient to adopt the classical negation in order to define the possibility operator.

The following are then defined symbols in the language of  $\mathbf{mbC}^{k,l,m,n}$ :

1.  $\sim\alpha \stackrel{\text{def}}{=} \alpha \supset [\beta \wedge (\circ\beta \wedge \neg\beta)]$ , for a fixed  $\beta$  (Classical negation)
2.  $\diamond\alpha \stackrel{\text{def}}{=} \sim\Box\sim\alpha$  (Possibility operator)

**Definition 2.1.** Let  $\mathbf{2} \stackrel{\text{def}}{=} \{0,1\}$  be the set of truth-values, where 1 denotes the ‘true’ value and 0 the ‘false’ value. An  $\mathbf{mbC}$ -valuation is a function  $v : \text{For}_{\mathbf{mbC}} \rightarrow \mathbf{2}$  subjected to the following clauses:

- (v1)  $v(\alpha \wedge \beta) = 1$  iff  $v(\alpha) = 1$  and  $v(\beta) = 1$
- (v2)  $v(\alpha \vee \beta) = 1$  iff  $v(\alpha) = 1$  or  $v(\beta) = 1$
- (v3)  $v(\alpha \supset \beta) = 1$  iff  $v(\alpha) = 0$  or  $v(\beta) = 1$
- (v4)  $v(\neg\alpha) = 0$  implies  $v(\alpha) = 1$
- (v5)  $v(\circ\alpha) = 1$  implies  $v(\alpha) = 0$  or  $v(\neg\alpha) = 0$

By an immediate extension of the clauses in Definition 2.1, we define the *modal valuations* for  $\mathbf{mbC}^{k,l,m,n}$  by adding:

- (v6)  $v(\Box\alpha, w) = 1$  iff  $v(\alpha, w') = 1$ , for all  $w'$  such that  $wRw'$

In the following we define Kripke models for  $\mathbf{mbC}^{k,l,m,n}$  based on  $\mathbf{mbC}$ -valuations. This model is able to provide a completeness result for  $\mathbf{mbC}^{k,l,m,n}$  based on non-trivial maximal sets (cf. [BS]).

**Definition 2.2.** A Kripke model  $\mathfrak{M}$  for  $\mathbf{mbC}^{k,l,m,n}$  is a triple  $\langle W, R, v \rangle$ , where:

- (i)  $W \neq \emptyset$ ;
- (ii)  $R$  satisfies the convergent property;
- (iii)  $v$  is a  $\mathbf{mbC}^{k,l,m,n}$ -valuation.

In an analogous way, it is possible to define a Kripke model also for systems  $\mathbf{bC}^{k,l,m,n}$  and  $\mathbf{Ci}^{k,l,m,n}$  by adding the convenient valuations that corresponds to each system.

Based upon an adaption of the classical Lindenbaum construction and in the notion of saturated canonical models, it is possible to show that  $\mathbf{mbC}^{k,l,m,n}$  is sound and complete with respect to the class of corresponding Kripke models.

**Teorema 2.3.** (Completeness)

Any logic in the family  $\mathbf{mbC}^{k,l,m,n}$  is sound and complete with respect to the Kripke models for  $\mathbf{mbC}^{k,l,m,n}$ .

This result can be extended also for  $\mathbf{bC}^{k,l,m,n}$  and  $\mathbf{Ci}^{k,l,m,n}$  with a little additional effort, by using substantially the same strategy as for  $\mathbf{mbC}^{k,l,m,n}$  (cf. [BS]).

### 3 Modal possible-translations semantics

Now, we present a formal definition of a modal possible-translations semantic structure adequate for  $\mathbf{mbC}^{k,l,m,n}$ . As we said, the definition for other catholic systems mentioned here is analogous. We start from the standard definition of possible-translations semantics given for  $\mathbf{mbC}$ , cf. [CCM07].

In our case, it will be convenient to adopt a slightly different definition of possible-translations structure, working with a unique traduct whose signature may contains copies of a given connective (so, for example, more than one negation). In this way, in this signature we may adopt only one kind of modal operator, so the modal component of the traduct remains constant. This will be clear when we expose the clauses of translations.

Before introducing the formal definition of possible-translations semantics it is convenient to consider the matrices cf. [CCM07] which comprise the environment that interprets all possible-translations of each formula. These matrices are also used to define the modal possible-translations semantics for the systems  $\mathbf{mbC}^{k,l,m,n}$ .

Let  $\mathcal{A} = \langle \{F, t, T\}, \wedge, \vee, \rightarrow, \neg_w, \neg_s, \circ_w, \circ_s \rangle$  be a  $\Sigma$ -algebra, where the operations are determined by the truth-tables below and let  $\mathcal{M} = \langle \mathcal{A}, \{t, T\} \rangle$  be a  $\Sigma$ -matrix, where  $T$  and  $t$  are distinguished truth-values in  $\mathcal{A}$ :

$\wedge$	$T$	$t$	$F$
$T$	$t$	$t$	$F$
$t$	$t$	$t$	$F$
$F$	$F$	$F$	$F$

$\vee$	$T$	$t$	$F$
$T$	$t$	$t$	$t$
$t$	$t$	$t$	$t$
$F$	$t$	$t$	$F$

$\supset$	$T$	$t$	$F$
$T$	$t$	$t$	$F$
$t$	$t$	$t$	$F$
$F$	$t$	$t$	$t$

  

	$\neg_w$	$\neg_s$	$\circ_w$	$\circ_s$
$T$	$F$	$F$	$t$	$F$
$t$	$F$	$t$	$F$	$F$
$F$	$T$	$t$	$t$	$F$

The *translations* from  $\mathbf{mbC}$  to the 3-valued logics are all the functions in the set  $Tr$  of mappings  $* : For_{mbC} \rightarrow For_{\mathcal{M}}$  subjected to the following restrictive clauses:

(tr0)  $p^* = p$ , for  $p$  a propositional variable;

(tr1)  $(\alpha \# \beta)^* = \alpha^* \# \beta^*$ , for all  $\# \in \{\wedge, \vee, \supset\}$ ;

(tr2)  $(\neg\alpha)^* \in \{\neg_w\alpha^*, \neg_s\alpha^*\};$

(tr3)  $(\circ\alpha)^* \in \{\circ_w\alpha^*, \circ_s\alpha^*, \circ_w(\neg\alpha)^*\}.$

The pair  $PT = \langle \mathcal{M}, Tr \rangle$  is a *possible-translations semantical structure* for  $\mathbf{mbC}$ . If  $\vDash_{\mathcal{M}}$  denotes the semantical consequence relation in  $\mathcal{M}$ , and  $\Gamma \cup \{\alpha\}$  is a set of formulas of  $\mathbf{mbC}$ , the associated *PT-consequence relation*,  $\vDash_{PT}$  is defined by setting:

$$\Gamma \vDash_{PT} \alpha \quad \text{iff} \quad \Gamma^* \vDash_{\mathcal{M}} \alpha^* \quad \text{for all translations } * \text{ in } Tr.$$

Any image of some mapping in  $Tr$  is called a *possible-translation* of a formula. It is usual in the standard PTS to think that a formula is being translated into different logics, since it is usual to work with a set of 3-valued logics that interpret the connectives in different ways. So, in order to decide whether any formula is a theorem of  $\mathbf{mbC}$  we need to check that its translations will be theorems within each logic considered as a traduct.

For our purposes, however, it will be convenient to stipulate that each formula is translated not into a formula only, but into a set of formulas. This will have the effect that we can deal with a unique traduct, instead of a bunch of logics. In this way, each connective can be interpreted as a family of 3-valued connectives. So any formula will be translatable into a set of formulas, which will be the set of all possible-translations of this formula<sup>2</sup>. The validity of a formula continues to be dependent on all translations as before, but the checking is made in a unique 3-valued traduct.

To define a modal possible-translations semantics for  $\mathbf{mbC}^{k,l,m,n}$  we need to define a  $Tr_M$  as a  $Tr$  augmented by the following restriction:

(tr4)  $(\Box\alpha)^* = \Box\alpha^*$

The (modal) traduct is now a 3-valued modal logic, in the style, e.g., of [Fit91] and [Fit92]. The validity of formulas of the form  $\Box\alpha$  will be defined in the modal traduct as:

$$v_{\mathcal{M}}(\Box\alpha, w) \in \{T, t\} \quad \text{iff} \quad v_{\mathcal{M}}(\alpha, w') \in \{T, t\} \quad \text{for all } w' \text{ such that } wRw'$$

It is interesting to observe that clause *tr4* given above does make sense, because now the translation of any formula goes from  $For$  to the power-set  $\wp(For)$ , where  $For$  represents the set of formulas of  $\mathbf{mbC}^{k,l,m,n}$ . This shows that the task of interpreting the modal character of  $\mathbf{mbC}^{k,l,m,n}$  is transferred to the much simpler 3-valued modal traduct, and this is quite natural since the modal traduct of  $\mathbf{mbC}^{k,l,m,n}$  is defined in an analogous way as  $\mathbf{mbC}^{k,l,m,n}$  itself, i.e., the accessibility relation satisfies the same property.

**Definition 3.1.** *A modal possible-translations semantics for  $\mathbf{mbC}^{k,l,m,n}$  is a quadruple  $\langle \mathcal{M}, Tr_M, W, R \rangle$ , where  $\langle \mathcal{M}, Tr_M \rangle$  is a PTS for  $\mathbf{mbC}^{k,l,m,n}$  and  $\langle W, R \rangle$  is a frame for  $\mathbf{mbC}^{k,l,m,n}$ .*

<sup>2</sup>A somewhat similar approach, in categorial terms, is done in [dSR07], in which translations as multi-functions are considered instead of functions.

As expected, the objective of this kind of semantics is to provide interpretations to the connectives of  $\mathbf{mbC}^{k,l,m,n}$  with respect to 3-valued translations, and in this case it is convenient to read the truth-value  $t$  as “truth by default” while  $T$  and  $F$  means respectively “true” and “false”.

Now the proof of characterization of  $\mathbf{mbC}^{k,l,m,n}$  (soundness and completeness) with respect to modal possible-translations semantics follows from two steps, whose proof is easily adapted from the propositional case:

- The proof of soundness is obtained by checking that the finite collection of axioms of  $\mathbf{mbC}^{k,l,m,n}$  are 3-valued tautologies, and that all possible-translations of  $\mathbf{mbC}^{k,l,m,n}$ -rules preserve 3-valued tautologies in  $\mathcal{M}$ . This is easy (but tedious).
- The proof of completeness is obtained by showing that each  $\mathbf{mbC}^{k,l,m,n}$ -valuation  $v$  of a Kripke model for  $\mathbf{mbC}^{k,l,m,n}$  determines a translation  $*$  and a 3-valued valuation defined over the truth-tables of  $\mathcal{M}$ , denoted by  $v_{\mathcal{M}}$ , such that for every formula  $\alpha$  of  $\mathbf{mbC}^{k,l,m,n}$  it holds:

$$v_{\mathcal{M}}(\alpha^*) \in \{t, T\} \quad \text{iff} \quad v(\alpha) = 1$$

The proof here is an easy extension of theorems 67 and 68 of [CCM07], taking care of extending the complexity measure in definition 62 by adding:  $l(\Box\alpha) = l(\alpha) + 1$ .

In order to illustrate the intuition of controlled explosion in cathodic system, consider situations where worlds  $w_1$  to  $w_4$  contain the following sentences:

$$\begin{aligned} w_1 &= \{\Box\alpha, \neg\Box\alpha, \circ\Box\alpha\} \\ w_2 &= \{\Box\alpha, \Box\neg\alpha, \Box\circ\alpha\} \\ w_3 &= \{\Box\alpha, \neg\Box\alpha, \Box\circ\alpha\} \\ w_4 &= \{\Box\alpha, \Box\neg\alpha, \circ\Box\alpha\} \end{aligned}$$

In  $w_1$  the explosion occurs immediately, independent of any property of the accessibility relation associated to the model. In the case of  $w_2$  explosion occurs in each world  $w'$  such that  $w_2 R w'$ , i.e., the explosion *only* does not occur only in frames where  $w_2$  is either an end point or an isolated point. In the other cases the explosion will occur or not, depending on the properties of the modal frame associated.

The concept of “impossible world” or “non-standard world” is imagined as a way to model certain logic notions that cannot be adequately expressed by means of usual possible worlds. Sometimes impossible worlds are considered a pair with “non-normal worlds”, and they are important for hypothetically invalidating certain logic laws, or to separate logical laws. For instance, Zalta in [Zal97], proposes a theory of what he calls “genuine impossible worlds”, and not a theory of *ersatz* impossible worlds. The models of our paraconsistent modal logics, in special the models described by possible-translations semantics, are very suitable candidates for “real impossible worlds”, in the sense that

Zalta defends. Such models are not *ersatz* in the precise sense that they are mathematical objects and one can make logic deductions with them.

If we understand the notion of *impossible world* as a world that admits contradiction without exploding, then  $w_3$  is an example of an impossible world and  $w_4$  is a world which can be related to an impossible world in sense that, for each world  $w''$  such that  $w_4 R w''$ ,  $w''$  is a impossible world since  $\alpha$  and  $\neg\alpha$  belong to  $w''$ . In this way, the machinery of cathodic logics is an apt tool to express notions of impossible worlds or even to explain how impossible worlds can be accessed.

In [Nol97], Nolan argues that there is a variety of areas where it may be very useful to reason with impossibilities in a deductively non-trivial way, but he also argues that to modify the notion of logical consequence to accommodate impossible situations is a mistake, as impossibilities can survive in classical reasoning. Now, what our paraconsistent modal systems do is just like that- we enlarge the notion of logical consequence, but do not throw away any “classical” modal reasoning, as such reasoning can be recovered within our systems.

As argued in [CCM07], possible-translations semantics offer an immediate decision procedure for any system that is complete with respect to a possible-translations semantical structure  $PT = \langle \mathcal{M}, Tr \rangle$ , where  $\mathcal{M}$  is decidable and  $Tr$  is recursive. In this way, our semantics will easily provide a decision procedure for the logics treated here.

Similar semantics as we have presented here seem to be easily adapted to the non-deterministic semantics given by A. Avron (see e.g. [Avr07]) for families of logics related to LFIs (although, as far as we know, Avron has not considered modal logics).

An approach for algebraizing LFIs based on an idea similar to that of a possible-translations structure was presented in [BSCC07] and [BSC05]. What our semantics would represent for the corresponding (hard) problem of algebraizing modal paraconsistent logics is a question still to be investigated.

## References

- [Avr07] A. Avron. Non-deterministic semantics for families of paraconsistent logics. In D. Gabbay J.-Y. Béziau, W. A. Carnielli, editor, *Handbook of Paraconsistency*, pages 285–320, England, 2007. College Publications, London.
- [BS] J. Bueno-Soler. Multimodalidades anódicas e catódicas: a negação controlada em lógicas multimodais e seu poder expressivo, (in Portuguese). Ph.D. Thesis under development, Unicamp.
- [BS07] J. Bueno-Soler. Extensões modais das lógicas **mbC** e **Ci**. In C. A. Mortari and L. H. de A. Dutra, editors, *Proceedings of Principia–V Simpósio Internacional Principia*, pages 99–101, Florianópolis, SC, 2007. NEL/UFSC.

- [BSC05] J. Bueno-Soler and W. A. Carnielli. Possible-translations algebraization for paraconsistent logics. *Bulletin of the Section of Logic*, 34(2):77–92, 2005. Preprint available at *CLE e-Prints*, vol. 5, n. 6, 2005. <http://www.cle.unicamp.br/e-prints/vol\5,n\6,2005.html>.
- [BSCC07] J. Bueno-Soler, M. E. Coniglio, and W. A. Carnielli. Possible-translations algebraizability. In D. Gabbay J.-Y. Béziau, W. A. Carnielli, editor, *Handbook of Paraconsistency*, pages 321–340, England, 2007. College Publications, London.
- [CC07] W. A. Carnielli and M. E. Coniglio. Bridge principles and combined reasoning. In T. Müller and A. Newen, editors, *Logik, Begriffe, Prinzipien des Handelns*, pages 32–48. Mentis Verlag, Mentis Verlag, 2007.
- [CCM07] W. A. Carnielli, M. E. Coniglio, and J. Marcos. Logics of formal inconsistency. In D. Gabbay and F. Guentner, editors, *Handbook of Philosophical Logic*, volume 14, pages 1–93, Amsterdam, 2007. Springer-Verlag.
- [CL03] A. Costa-Leite. Paraconsistência, modalidades e cognoscibilidade. Master’s thesis, IFCH-Unicamp, Campinas, SP, Brasil, 2003.
- [CM02] W. A. Carnielli and J. Marcos. A taxonomy of **C**-systems. In W. A. Carnielli, M. E. Coniglio, and I. M. L. D’Ottaviano, editors, *Paraconsistency - the Logical Way to the Inconsistent*, volume 228 of *Lecture Notes in Pure and Applied Mathematics*, pages 1–94, New York, 2002. Marcel Dekker.
- [Con07] M. E. Coniglio. Logics of deontic inconsistency. *CLE e-Prints*, 7(4), 2007.
- [CP08] W. A. Carnielli and C. Pizzi. *Modalities and Multimodalities*. Springer, 2008.
- [dCC86] N. C. A. da Costa and W. A. Carnielli. On paraconsistent deontic logic. *Philosophia - the Philosophical Quarterly of Israel*, 16(3/4):293–305, 1986.
- [dSR07] T. de Souza Reis. Representação de lógicas por traduções possíveis: um formalismo de decomposição de lógicas. In C. A. Mortari and L. H. de A. Dutra, editors, *Proceedings of Principia-V Simpósio Internacional Principia*, page 165, Florianópolis, SC, 2007. NEL/UFSC.
- [Fit91] M. Fitting. Many-valued modal logics. *Fundamenta Informaticæ*, 15:235–254, 1991.

- [Fit92] M. Fitting. Many-valued modal logics. *Fundamenta Informaticæ*, 17:55–73, 1992.
- [Hen49] L. Henkin. Fragments of the propositional calculus. *The Journal of Symbolic Logic*, 14(1), 1949.
- [HPvdT07] J. Hansen, G. Pigozzi, and L. van der Torre. Ten philosophical problems in deontic logic. In G. Boella, L. van der Torre, and H. Verhagen, editors, *Normative Multi-agent Systems*, number 07122 in Dagstuhl Seminar Proceedings, Dagstuhl, Germany, 2007. Internationales Begegnungs- und Forschungszentrum fuer Informatik (IBFI), Schloss Dagstuhl, Germany. <http://drops.dagstuhl.de/opus/volltexte/2007/941/>.
- [LS77] E. J. Lemmon and D. Scott. An introduction to modal logic. In K. Segerberg, editor, *The Lemmon Notes*. Oxford, Blackwell, 1977.
- [Nol97] D. Nolan. Impossible worlds: A modest approach. *Notre Dame J. Formal Logic*, 38(4):535–572, 1997.
- [PC] N. M. Peron and M. E. Coniglio. Logics of deontic inconsistencies and paradoxes. CLE e-prints, volume 8(6), 2008. [http://www.cle.unicamp.br/e-prints/vol\\\_8,n\\\_6,2008.html](http://www.cle.unicamp.br/e-prints/vol\_8,n\_6,2008.html).
- [Seg67] K. Segerberg. Some modal logics based on a three-valued logic. *Theoria*, 33(4):53–71, 1967.
- [vF72] B. C. van Fraassen. The logic of conditional obligation. *Journal of Philosophical logic*, 70(1):417–438, 1972.
- [vF73] B. C. van Fraassen. Values and the heart’s command. *Journal of Philosophy*, 70(1):5–19, 1973.
- [Zal97] E. N. Zalta. A classically-based theory of impossible worlds. *Notre Dame J. Formal Logic*, 38(4):640–660, 1997.